Problem 570: A 30-year Ramsey Riddle Finally Clicks

A juicy, mass-friendly retelling for readers with a general mathematics background.

Setting the stage.

Imagine you have N points, and you join every pair by an edge—the "everyone-knows-everyone" graph K_N. Now you paint each edge either **red** or **blue**. Ramsey theory is the art of proving that if N is big enough, *some* unavoidable pattern must appear, no matter how mischievous your coloring strategy is.

Problem 570 asks for the smallest N that forces one of two outcomes:

- a **red** cycle C_k (a loop of length k), or
- a **blue** copy of some graph H that has m **edges** (and no isolated vertices).

That threshold is the Ramsey number $R(C_k, H)$.

The punchline conjecture (Erdos-style).

For large enough m (depending on k), the "worst-case" H was expected to behave like a **matching**: mK_2, i.e., m disjoint edges—the most "spread-out" way to spend m edges.

And the conjectured exact formula was delightfully clean:

$$R(C_k, H) \;=\; 2m \;+\; \left\lfloor \frac{k-1}{2} \right\rfloor .$$

This had been proved for **all even** k and for $k = 3$; later digging revealed $k = 5$ was also already settled. So the stubborn zone was: **odd** k **beyond** 5.

Why even cycles behave nicely, but big odd cycles bite back.

Here is the intuition, stripped down to the engine.

1) Even cycles are bipartite—and that changes the rules.

If k is even, C_k is bipartite. In extremal graph theory, forbidding bipartite graphs tends to force strong sparseness patterns. Roughly: if red must avoid such a cycle, red cannot be too dense in the wrong way, so blue becomes dense enough to contain the desired H.

2) Small odd cycles are local; big odd cycles are global.

A red triangle C_3 is tiny: you can hunt it by zooming in around a single vertex. A C_5 is still small enough that neighborhood tricks can work.

But for odd $k > 5$, that cozy logic breaks. The red graph can be structured to dodge short-cycle traps while still being complicated enough to resist forcing a blue H. The proof needs a global scaffold, not just local policing.

A key global idea: long red roads and a well-placed shortcut.

Suppose you sit exactly at the conjectured threshold

$$N = 2m + \left\lfloor \frac{k-1}{2} \right\rfloor,$$

and imagine the nightmare scenario: *no red C_k and no blue H*. Then the red graph is still forced to contain a *slightly longer* cycle C_{k+1}. In particular, red must contain **long paths**—think of them as long, non-branching "roads."

The strategy becomes:

> Find a long red path, and then locate a vertex that connects into that path by two red edges—a shortcut that closes the loop and creates the forbidden red C_k.

Where the solution came from.

A successful attack was developed by Cambie, Freschi, Morawski, Petrova, and Pokrovskiy (work carried out during 2025–2026 and later uploaded). The collaboration grew out of a virtual Ramsey-theory working session of the Sparse Graph Coalition—basically a research hackathon, but for serious graph theory.

What the proof does (without drowning you in notation).

Fix a red/blue coloring of K_N and assume you are in the nightmare scenario: no red C_k and no blue H. Then you push until the assumptions contradict themselves.

Some cases can be dispatched with existing tools:

- If H is disconnected, you would like to find its components one by one in blue—but naive induction can fail because removing a single K_2 component does not improve the key density ratio. That is why the matching case needs special care.
- Extremely dense or extremely sparse components can be handled using known bounds.

So the fight concentrates on the middle-density regime, where new ideas are needed—and those ideas revolve around forcing the red cycle.

A three-zone lens around one vertex.

Pick a vertex v. Now classify everyone else by how they connect to v via red edges:

- U_1: vertices joined to v by a **red** edge.
- Π: vertices not in U_1, but reachable from v by a two-step red walk $v - u - x$ with both edges red (so $u \in U_1$ and ux is red).
- U_2: all remaining vertices.

Now come two wonderfully direct "if you see this, you are done" triggers:

- If U_1 contains a red path P_{k-1} (a red chain on $k-1$ vertices), then adding v closes it into a red C_k.
- If Π contains a long enough red path—specifically P_{2k}—then a technical lemma lets you assemble a red C_k from it as well.

Why paths? Because paths are bipartite, and bipartite patterns are notoriously "harder to forbid." That means long red paths tend to appear unless blue is already rich enough to contain H.

The final squeeze.

The proof then runs a carefully tuned induction that says, in effect:

- either U_1 must reveal a long red path,
- or Π must reveal a long red path,
- or else the blue edges are structured densely enough that a blue H must appear.

Any route contradicts the nightmare scenario. That pins down the conjectured Ramsey number at the exact value $2m + \lfloor (k-1)/2 \rfloor$ (for large enough m in terms of k).

Source note. This is a popularized reformulation of the blog-style discussion at erdosproblems.com/forum/thread/blog:3.